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# A REVIEW ON ABSTRACT SPACES AND FIXED POINT THEOREM

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## ABSTRACT

We discuss a brief introduction of cone metric space where the set of real number $\mathbb{R}$ is replaced by real Banach space $\boldsymbol{E}$, partial rectangular metric space, partial-$b$-metric space etc. In each of these space several changes have been made or added in the properties of metric space which generalized the famous Banach contration principle.

***Keywords-*** *Cone metric space, partial rectangular metric space, partial b – metric space etc., fixed point theorem.*

## I. INTRODUCTION

Let $X$ be a non-empty set and $T: X \to X$ be a mapping. An element $x^* \in X$ is called a fixed point of mapping $T$, if it is invariant under the mapping $T$, i.e., $T(x^*) = x^*$. In Mathematics, a fixed point theorem is a result ensuring existence or uniqueness or both of a self-map on a non-empty set.Fixed point theory was initiated by the French mathematician Poincare in (1854-1912). The foundation of fixed point approaches towards the solution of the problems concerned to the mathematical analysis.In 1906, the famous French mathematician Frechet [5] introduced the concept of a metric space (a notion of distance) which is as follows:

Let $X$ be a non-empty set and $d : X \times X \to \mathbb{R}$ where $\mathbb{R}$ the set of reals be a function. Then $d$ is called a metric on $X$ and the pair $(X, d)$ is called a metric space if the following conditions are satisfied:

- $0 \le d(x, y)$ and $d(x, y) = 0$ iff $x = y$.
- $d(x, y) = d(y, x)$.
- $d(x, y) \le d(x, z) + d(z, y) \quad \forall\, x, y, z \in X$.

In 1922, Banach proved the well-known result ‘Banach fixed point theorem’ in the setting of metric space which is also known as ‘Banach Contraction Principle’ to establish the existence of the solution for non-linear operator equations and the integral equations. It states that:

**Banach Contraction Principle -** Let$(X, d)$ be a complete metric space and $T : X \to X$ be a contraction, satisfying:
$d\big(T(x), T(y)\big) \le \lambda\, d(x, y)$, for $\lambda \in [0, 1)$and $\forall\, x, y \in X$

Then, there exists a point $x^* \in X$ such that $T(x^*) = x^*$.Furthermore, for every $x_0 \in X$, the iterative sequence $\{x_n\}$ defined by $x_n = T(x_{n-1}),\ \forall\, n \in X$ converges to$x^*$.

Since then, because of simplicity and usefulness of this result it has become a very popular tool in solving the existence problems in many branches of mathematical analysis. Further work is done on different metric spaces and various type of mappings defined on metric space. To generalize the results of metric spaces different type of spaces, such as cone metric space, partial metric space, rectangular metric space, $b$-metric spaces, partial $b$-metric spaces are defined on sets as the requirement of applications and theories. The definitions of these spaces are as follows:

**Cone metric space [6]:** A cone metric space is an ordered pair $(X, d)$, where $X$ is a non-empty set and $d : X \times X \to E$ is a mapping satisfying for all $x, y, z \in X$,
$i)$ $d(x, y) \in P$, i.e., $\theta \le d(x, y)$, and $d(x, y) = \theta$ iff $x = y$;
ii) $d(x, y) = d(y, x)$;
$iii)$ $d(x, y) = d(x, z) + d(z, y)$.

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**Example:** Let $E = \mathbb{R}^2, P = \{(x, y) \in E : x, y \geq 0\}$,
$X = \mathbb{R}$ *and* $d: X \times X \to E$ *such that*
$d(x, y) = (|x - y|, \alpha|x - y|)$,
*where* $\alpha \geq 0$ is a constant. Then $(X, d)$ is a cone metric space.

**Partial Metric Space [12]:** A Partial Metric Space is an ordered pair $(X, \rho)$ where $X$ is any non-empty set and $\rho : X \times X \to \mathbb{R}$ is a mapping satisfying for all $x, y, z \in X$,
i) $\rho(x, y) \geq 0$;
ii) $x = y$ iff $\rho(x, x) = \rho(x, y) = \rho(y, y)$;
iii) $\rho(x, x) \leq \rho(x, y)$;
iv) $\rho(x, y) = \rho(y, x)$;
v) $\rho(x, y) \leq \rho(x.z) + \rho(z, y) - \rho(z, z)$;

**Rectangular Metric Space [3]:** A Rectangular Metric Space is an ordered pair $(X, d)$ where $X$ is any non-empty set and $d: X \times X \to \mathbb{R}$ a mapping satisfying for all $x, y, z \in X$,
i) $0 \leq d(x, y)$and iff $x = y$;
ii) $d(x, y) = d(y, x)$;
iii) $d(x, y) \leq d(x, w) + d(w, z) + d(z, y)$for all $x, y \in X$;and for all distinct point $w, z \in X - \{x, y\}$ (rectangular property).

**Partial Rectangular Metric Space [14]:** A Partial Rectangular Metric Space is an ordered pair $(X, \rho)$ where $X$ is any non-empty set and $\rho : X \times X \to \mathbb{R}$ is a mapping satisfying for all $x, y, z \in X$,
*i)* $\rho(x, y) \geq 0$;
*ii)* $x = y$ iff $\rho(x, x) = \rho(x, y) = \rho(y, y)$;
iii) $\rho(x, x) \leq \rho(x, y)$;
iv) $\rho(x, y) = \rho(y, x)$;
v) $\rho(x, y) \leq \rho(x, w) + \rho(w, z) + \rho(z, y) - \rho(w, w) - \rho(z, z) \ \forall\, x, y \in X$ andfor all distinct points $w, z \in X - \{x, y\}$.

**Partial $b$-metric space [13]:** A Partial $b$ −metric space is an ordered pair $(X, b)$ where $X$ is any non-empty set and $b: X \times X \to \mathbb{R}^+$ is a mapping satisfying for all $x, y, z \in X$,
i) $x = y$iff $b(x, x) = b(x, y) = b(y, y)$;
ii) $b(x, x) \leq b(x, y)$;
iii) $b(x, y) = b(y, x)$;
iv) There exists a real number $s \geq 1$ such that

$$b(x, y) \leq s[b(x, z) + b(z, x)] - b(z, z);$$

## II. REVIEW OF LITERATURE

The study of metric space was imitated by the famous French mathematician Frechet [5] in 1906. After that, in 1922 the Banach contraction principle was established for metric space. After this, several mathematician generalized the concept of metric space in different direction. The Banach contraction principle demands the continuity of the mapping which is a very strong condition and cannot hold in some practice problems. Kannan [10] proved the existence of a fixed point for a map that can have a discontinuity in a domain, however the maps involved in every case were continuous at the fixed point.

In 1976, Jungck [8] proved a common fixed point theorem for commuting maps, generalizing Banach contraction principle. This theorem has applications in solving and obtaining the common solutions of systems of equations and in the other problems involving common solutions. But the result suffer from one drawback that it require the continuity of one of the two maps involved.

In 1994, Matthews [12] introduced the partial metric space with the property that self-distance of any point may not be zero. This was useful for the study of denotational semantics of dataflow network. Further Matthews [12] showed that the Banach contraction principle is valid in partial metric space and be applied in program verification. In this sequel, Branciari [3], introduced rectangular metric spaces by replacing triangular inequality which involves four or more points instead of three and improved Banach contraction principle

In 2007, Huang and Zhang [6] initiated the concept of cone metric space as a generalization of metric spaces and proved some topological properties and some fixed point results. An example in it shows that the class of contraction mappings in cone metric spaces is much wider than that in a usual metric space. Azam et al. [1] extended the concept of rectangular metric spaces initiated by [3] to cone rectangular metric space by replacing the set of real numbers of a rectangular metric space to an ordered Banach space and proved Banach contraction principle. Jain et al. [7] proved some common fixed point results for weakly compatible pairs in these space by removing the normality of cone. Beg et al. [2] and Du [4] generalized this approach by using cones in topological vector spaces (TVS) instead of Banach space. Kadelburg et al. [9] developed further theory of TVS-cone metric space using Minkowskifuntionals and established the equivalence between some fixed point results in metric space and in TVS-cone metric space. Thus a lot of results in cone metric setting can be directly obtained from their metric counterparts. Their approach was easier than that of Du [4].

In 2013, Shukla [13] introduced the concept of partial *b*-metric spaces as a generalization of partial metric and *b*-metric spaces. In 2014, Shukla [14] also introduced the concept of partial rectangular metric spaces as a generalization of rectangular metric and partial metric spaces and some properties of partial rectangular metric spaces. Liu and Xu [11], used the Banach algebra instead the Banach spaces as a codomain of the cone metric and introduced the cone metric spaces with Banach algebra. They also showed that the fixed point results in such spaces cannot be derived by the corresponding fixed point results of usual metric spaces.

## III. CONCLUSION

In a wide range of mathematical problems the existence of a solution is equivalent to obtain a fixed point for a suitable map. The existence of a fixed point is therefore of paramount importance in several areas of mathematics and other sciences. Fixed point results provide conditions under which problems have solution. The theory itself is a beautiful mixture of analysis (pure and applied) topology and geometry. Fixed point theory has many applications such as in root finding, establishing the existence and uniqueness of solutions, for solving integral equations. In particular, fixed point techniques have been applied in such diverse field as biology, chemistry, economics, engineering , game theory and physics.

The above discussion concludes that as per the need and the requirement various new concepts and new results have been developed which removed drawbacks of the existing concepts. The out coming results are more useful than the previous one which generalized Banach contraction principle. Our aim for reviewing this papers is to make a convenient way to getting history of this field and what work is done in this area for the researchers so that they can get benefit from this in their work.

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